B. Sc. Part II (Hons)

4th Paper

Vector calculus

Differentiation of vectors

Let $\vec{F}$ be a vector function of a single scalar variable t.

i.e. $\vec{F} = \vec{F}(t)$.

$$\therefore \vec{F} = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}$$

$$\therefore \text{Derivative of } \vec{F}(t) = \frac{d}{dt}\left[\vec{F}(t)\right] = \vec{F}'(t)$$

Also, $$\frac{d}{dt}(\vec{r}_1 \cdot \vec{r}_2) = \frac{d\vec{r}_1}{dt} \cdot \vec{r}_2 + \vec{r}_1 \cdot \frac{d\vec{r}_2}{dt}$$

$$\frac{d}{dt}(\vec{r}_1 \times \vec{r}_2) = \frac{d\vec{r}_1}{dt} \times \vec{r}_2 + \vec{r}_1 \times \frac{d\vec{r}_2}{dt}$$

$$\frac{d}{dt}(\vec{r}_1 + \vec{r}_2) = \frac{d\vec{r}_1}{dt} + \frac{d\vec{r}_2}{dt}.$$

Q. If $\vec{r} = \vec{a}\cos\omega t + \vec{b}\sin\omega t$, prove that

$$\vec{r} \times \frac{d\vec{r}}{dt} = \omega \vec{a} \times \vec{b} \text{ and}$$

$$\frac{d^2\vec{r}}{dt^2} = -\omega^2 \vec{r},$$ $\vec{a}$ and $\vec{b}$ being constant vectors and ω is also a constant.

Proof Given $\vec{r} = \vec{a}\cos\omega t + \vec{b}\sin\omega t \quad \text{---} (1)$

Differentiating with respect to t, we get

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left(\vec{a}\cos\omega t + \vec{b}\sin\omega t\right)$$

$$\Rightarrow \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(\vec{a}\cos\omega t\right) + \frac{d}{dt}\left(\vec{b}\sin\omega t\right)$$

$$\Rightarrow \frac{d\vec{r}}{dt} = \vec{a}\frac{d}{dt}(\cos\omega t) + \vec{b}\frac{d}{dt}(\sin\omega t)$$

$$\Rightarrow \frac{d\vec{r}}{dt} = -\omega\sin\omega t\,\vec{a} + \omega\cos\omega t\,\vec{b} \quad \text{---} (2)$$

$$LHS = \vec{r}\times\frac{d\vec{r}}{dt} = \left(\vec{a}\cos\omega t + \vec{b}\sin\omega t\right)\times \left(-\omega\sin\omega t\,\vec{a} + \omega\cos\omega t\,\vec{b}\right)$$

$$= -\omega\sin\omega t\cos\omega t\,\vec{a}\times\vec{a} + \omega\sin\omega t\cos\omega t\,\vec{b}\times\vec{b} - \omega\sin^2\omega t\,\vec{b}\times\vec{a} + \omega\cos^2\omega t\,\vec{a}\times\vec{b}$$

$$= 0 + 0 + \omega\sin^2\omega t\,\vec{a}\times\vec{b} + \omega\cos^2\omega t\,\vec{a}\times\vec{b}$$

$$= \omega\left(\sin^2\omega t + \cos^2\omega t\right)\vec{a}\times\vec{b} = \omega\{\vec{a}\times\vec{b}\} = RHS$$

differentiating (2) with respect to t, we get

$$\frac{d^2\vec{r}}{dt^2} = -\omega^2\cos\omega t\,\vec{a} + (-)\omega^2.\sin\omega t\,\vec{b}$$

$$= -\omega^2\left(\vec{a}\cos\omega t + \vec{b}\sin\omega t\right)$$

$$= -\omega^2\vec{r} \quad [\text{using } (1)]$$

$$\Rightarrow \frac{d^2\vec{r}}{dt^2} = -\omega^2\vec{r}$$